Q.1. A) $\lambda = 3, -2, 1$ Eigenvalues of $A^2 + 2I$ are $\lambda^2 + 2$.

i.e. $9 + 2 = 11$

$(-2)^2 + 2 = 6$

$1^2 + 2 = 3$

$\underline{(11, 6, 3)}$

(B) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Let $Q = X^T A X$.

$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \Delta_3 > 0$

$\Rightarrow$ This quadratic form is positive definite.

(C) $z - \frac{\pi}{6} = 0$

$z = \frac{\pi}{6} = \frac{3.14}{6} < 1$

$z = \pi/6$ lies inside $C : |z| = 1$

$\oint \frac{\sin^6 z}{z - \pi/6} dz = 2\pi i \left(\sin^6 \pi/6\right) = \frac{2\pi i}{2^6}$

$= \frac{\pi i}{32}$

(D) $F(x, y, y') = \frac{y'^2}{x^2}$ $\Rightarrow$ $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$\frac{\partial F}{\partial y'} = \frac{2y'}{x^2}$

$0 - \frac{d}{dx}\left(\frac{2y'}{x^2}\right) = 0$

$\frac{2y'}{x^2} = C \Rightarrow y' = \frac{Cx^2}{2}$

$y' = C_1 x^2$ $\left(\because \frac{C}{2} = C_1\right)$

$\therefore y = C_1 \frac{x^3}{3} + C_2$.

$\boxed{y = c' x^3 + C_2}$ $\boxed{\therefore c' = \frac{C_1}{3}}$

(E) $\nabla \cdot \bar{F} = 0 \Rightarrow \bar{F}$ is solenoidal

$\therefore (-z + y x - y x + z) = 0$

$\nabla \times \bar{F} = 0(-2x + 3y) - (3y - 2x) = 0$ $\therefore \bar{F}$ is irrotational

$\bar{F}$ is both solenoidal & irrotational.

(F) Stoke's theorem, $\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n} \, ds$

$\nabla \times \bar{F} = 0$

$\therefore \oint_C \bar{F} \cdot d\bar{r} = 0$

Q.2. A) $\lambda = 1, 2, 3$ for $\lambda = 1$, $X_1 = (4, 3, 2)$
for $\lambda = 2$, $X_2 = (3, 2, 1)$
for $\lambda = 3$, $X_3 = (2, 1, 1)$

$A \cdot M = G \cdot M$

$$M = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(B) Let $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$

$\because y(0) = 0$, $c_0 = 0$

$y(1) = 0$, $c_1 + c_2 = 0$ $\boxed{c_2 = -c_1}$

$\bar{y}(x) = c_1 x(1-x)$

$\bar{y}'(x) = c_1(1-2x)$.

$$I = \int_0^1 (2x\bar{y} - \bar{y}^2 - \bar{y}'^2)\,dx = c_1\left(\frac{1}{6} - \frac{11}{30}c_1\right) = \frac{c_1}{6} - \frac{11 c_1^2}{30}$$

$$\frac{dI}{dc_1} = 0 \Rightarrow \frac{1}{6} - \frac{22}{30}c_1 = 0 \qquad c_1 = \frac{5}{22}$$

$$\bar{y} = \frac{5}{22}(x(1-x))$$

1C i) $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \Rightarrow A^3 - 5A^2 + 7A - 3I = 0$

$$A^2 = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix}$$

CH Theorem is verified.

ii) $I = \int_0^{1+i} (x - y + ix^2)\,(dz)$ Along the line $z=0$ to $z=1+i$

$y = x$
$dy = dx$

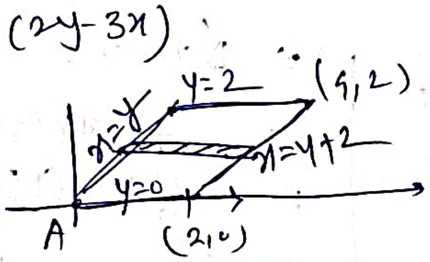
$$= \int_0^1 (\cancel{x} - \cancel{x} + ix^2)\,(dx + i\,dx)$$

$$= (1+i)\int_0^1 i\frac{x^3}{3}\,dx = \frac{i}{3}(1+i) = \frac{(i-1)}{3}$$

1D Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$P = 2x - 5y$
$Q = -(2y - 3x)$



$$= \iint_R (3+5)\,dx\,dy$$

$$= 8 \int_0^2 \int_y^{y+2} dy\,dx$$

$$= 8 \int_0^2 (y + 2 - y)\,dy$$

$= 16(x)_0^2$
$= 32$

SK2
(2)

$\boxed{E}$ $f(z) = \dfrac{2}{z^2-3z+2} = \dfrac{2}{(z-1)(z-2)} = \dfrac{-2}{z-1} + \dfrac{2}{z-2}$

(By partial fraction)

i) $|z|<1, \ |z|<2$

$f(z) = 2(1-z)^{-1} - (1-\frac{z}{2})^{-1}$

ii) $1<|z|<2 \qquad 1<|z|, \ |z|<2$

$f(z) = -\frac{2}{z}[1-\frac{1}{z}]^{-1} - [1-(\frac{z}{2})]^{-1}$

iii) $|z|>2, \quad |z|>1$

$|\frac{2}{z}|<1, \quad |\frac{1}{z}|<1$

$f(z) = -\frac{2}{z}(1-\frac{1}{z})^{-1} + \frac{2}{z}(1-\frac{2}{z})^{-1}$

Q.2. (A) i) $\nabla \times \bar{F} = 0 \qquad$ ii) $\bar{F} = \nabla \phi$.

$\dfrac{\partial \phi}{\partial x} = z^2 + 2x + 3y \qquad \dfrac{\partial \phi}{\partial y} = 3x + 2y + z \ , \ \dfrac{\partial \phi}{\partial z} = y + 2zx$

$\phi = z^2x + x^2 + 3xy + \psi_1 \ , \ \phi = 3xy + y^2 + zy + \psi_2, \ \phi = yz + z^2x + \psi_3$

$\Rightarrow \phi = z^2x + x^2 + 3xy + y^2 + zy$

Work done $= \phi_B - \phi_A = (3(4) + 9 + 0 + 0 + 0) - (1+1)$

$= 12 + 9 - 2$

$\underline{W = 19}$

Q.3. $\boxed{A}$ $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda = 1, 1.$

let $A^{50} = \alpha_1 A + \alpha_0$

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$

$1 = \alpha_1 + \alpha_0$ — ①

diff. w.r.t. λ

$50\lambda^{49} = \alpha_1 \qquad$ put $\underline{\lambda = 1}$

$\boxed{50 = \alpha_1}$

$\alpha_1 + \alpha_0 = 1$

$50 + \alpha_0 = 1$

$\boxed{\alpha_0 = -49}$

$\therefore A^{50} = 50A - 49I$

$A^{50} = \begin{bmatrix} 51 & 50 \\ -50 & -49 \end{bmatrix}$

SKS

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Q3. (B) $z^3(z+4)=0 \Rightarrow z=-4$ lies outside c.

$z=0$ lies inside c.

Residue at $z=0$, $\lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[\cancel{z^3} \cdot \frac{1}{\cancel{z^3}(z+4)}\right]$

$= \frac{1}{2!} \lim_{z\to 0} \frac{d}{dz}\left(\frac{-1}{(z+4)^2}\right) = \frac{-1}{2} \times \lim_{z\to 0} \frac{(-2)}{(z+4)^3}$

$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{(4)^3} = \frac{1}{64}$

$\oint f(z)\,dz = 2\pi i\left(\frac{1}{64}\right) = \frac{\pi i}{32}$

(C) $Q = X^T A X$ where $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ $\therefore \lambda = 6, 3, 2$

$X_1 = (1, -2, 1)$

$X_2 = (1, 0, -1)$

$X_3 = (1, 1, 1)$

$P = \left[\frac{X_1}{\|X_1\|} \; \frac{X_2}{\|X_2\|} \; \frac{X_3}{\|X_3\|}\right] = \begin{bmatrix} \frac{1}{\sqrt6} & \frac{-2}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt2} & 0 & \frac{-1}{\sqrt2} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{bmatrix}$

$P^T A P = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 6y_1^2 + 3y_2^2 + 2y_3^2$

(D) $\bar r \cdot \nabla\phi = 3\phi$

where $\nabla\phi = (3x^2-3yz)i + (3y^2-3xz)j + (3z^2-3xy)k$

$\nabla\cdot\bar F = \operatorname{div}\bar F = 6(x+y+z)$

(E) $F = y^2 - y'^2$

$y''+y=0$

$(D^2+1)y=0$

$D=\pm i$

$y = c_1\cos x + c_2 \sin x$

$y(0)=0$ $\boxed{c_1=0}$

$y\left(\frac{3\pi}{2}\right)=1 \Rightarrow c_2=-1$

$\boxed{y(x) = -\sin x}$

(F) $\nabla\cdot\bar F = 2x+y$

$\iint_S \bar F\cdot\hat n\,ds = \iiint_V (\nabla\cdot\bar F)\,dV = \iiint_V (2x+y)\,dx\,dy\,dz$

$= \int_0^1\int_0^1\int_0^1 (2x+y)\,dx\,dy\,dz = \int_0^1\int_0^1 (x^2+xy)\Big|_0^1\,dy\,dz$

$= \int_0^1\int_0^1 (1+y)\,dy\,dz = (1)\left(1+\frac{y^2}{2}\right)\Big|_0^1$

$= 1+\frac{1}{2}$

$= \frac{3}{2}$

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